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# Mean Convergence of Interpolation Polynomials in a Domain with Corners

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In this paper, we prove mean convergence of interpolation polynomials in a domain with some corners. 

## 1. INTRODUCTION

Let $D \subset \mathbb{C}$ be a domain bounded by a Jordan curve $\Gamma$, and let $U$ be the unit disk. $z = \Psi(w)$ denotes the one-to-one conformal mapping of $\mathbb{C} \backslash \bar{U}$ onto $\mathbb{C} \backslash \bar{D}$, normalized by the condition $\Psi(\infty) = \infty$, $\Psi'(\infty) > 0$. We denote the inverse mapping of $\Psi$ by $\Phi$.

Let

$$A(\bar{D}) = \{f: f \text{ analytic in } D \text{ and continuous on } \bar{D}\}.$$

In the case $\Gamma$ is rectifiable, let

$$\|f\|_p = \left\{\int_\Gamma |f(z)|^p \, |dz|\right\}^{1/p}. \tag{1.1}$$

For $S_n$ consisting of $n$ distinct points on $\Gamma$, $L_n(f, z)$ denote the Lagrange interpolation polynomials to $f \in A(\bar{D})$ on $S_n$. Generally, we cannot expect to find a sequence $\{S_n\}$ such that $L_n(f, z)$ is convergent uniformly on $\bar{D}$ for any $f \in A(\bar{D})$. Then it is reasonable to consider mean convergence of $\{L_n(f, z)\}$ on $\Gamma$. When $D = U$ and

$$S_n = \{e^{(2k\pi/n)i}, k = 1, 2, ..., n\}$$

it is well known [1] that

$$\lim_{n \to +\infty} \|f(z) - L_n(f, z)\|_p = 0 \tag{1.2}$$

for $0 < p < \infty$, $f \in A(\bar{D})$.



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In the general case, $S_n$ usually consists of the Fejér points on $\Gamma$, which means

$$S_n = \{\Psi(e^{(2k\pi/n)i}), k = 1, 2, ..., n\}.$$

Curtiss [2] showed (1.2) when $p = 2$ and $\Gamma$ is an analytic curve. Al'per and Kalinogorskaya [3] proved (1.2) when $\Gamma$ is $2 + \delta$ smooth. Recently, Shen and Zhong [4] got the same result when $\Gamma$ is $1 + \delta$ smooth. However, no corners are allowed in these papers. In [7], Thompson stated theorems for uniform convergence on closed subsets of $D$ that cover cases when $\Gamma$ possesses some corners with exterior angles not less than $\pi$. As we know, uniform convergence on closed subsets of $D$ is much weaker than mean convergence on $\Gamma$, and the restriction on exterior angles would lose some generalities. We refer the reader to [5, 6] for surveys for the problem and its history.

In this paper, we prove (1.2) in the case when $\Gamma$ possesses some corners. We call $D$ admissible, if there exist $\{r_j, j = 1, ..., K\} \subset \partial U$, $\{\alpha_j, j = 1, ..., K\} \subset (0, 2)$, $c_1 > 0$, $c_2 > 0$, and $\beta > 0$, such that

$$|\lambda(u)| \geqslant c_1, \qquad |u| \geqslant 1$$

and

$$|\lambda(u_1) - \lambda(u_2)| \leqslant c_2 |u_1 - u_2|^\beta, \qquad |u_1|, |u_2 \geqslant 1,$$

where

$$\lambda(u) = \Psi'(u) \prod_{j=1}^{k} (u^{-1} - \tau_j^{-1})^{1-\alpha_j}. \tag{1.3}$$

Clearly, if $D$ is admissible, $\Gamma$ posseses a continuously turning tangent except at the points $\Psi(\tau_j)$, $j = 1, ..., K$, at which $\Gamma$ has half tangents with exterior angles $\pi\alpha_j$. Conversely, if $\Gamma$ consists of a finite number of arcs with continuous curvature and the exterior angles not being $0, 2\pi$, then $D$ is admissible.

The main result in this paper is the following.

THEOREM. *Suppose that $0 < p < \infty$ and $D$ is admissible and that $S_n$ consists of the Fejér points. Then for any $f \in (A\bar{D})$,*

$$\lim_{n \to \infty} \|f(z) - L_n(f, z)\|_p = 0.$$

As in [5], the main idea of proof is using the theory of singular integral. First, we show that the Fejér points are uniformly separated inside a level curve. Second, we find a function $h$ to interpolate $f$, which may not be a polynomial but in analytic inside the level curve. Third, $L_n(f, z)$ is taken as the weighted singular integral of $h$. Finally, we show that the interpolation polynomial operators $L_n\colon A(\bar{D}) \to L^p$ are bounded uniformly by the theory of singular integral and the estimation of the weight.

In the following the domain $D$ is always assumed admissible, and $c_j$ denote positive constant only depending on $D$ and $p$.

## 2. Preliminary Facts

For $1 \leqslant |u|, |w| \leqslant 2$, we have [8, p. 387]

$$c_3^{-1} \leqslant \frac{|\Psi(u) - \Psi(w)|}{|u-w| \, (|u-\tau_k| + |u-w|)^{\alpha_k - 1}} \leqslant c_3, \tag{2.1}$$

where $\tau_k$ is the closest point to $u$ among $\{\tau_j, j = 1, ..., K\}$.

For $z \in \mathbb{C}$, $E \subset \mathbb{C}$, let

$$d(z, E) = \inf_{\zeta \in E} |z - \zeta|;$$

then for $\rho > 1$, we have

$$|\Psi(e^{it}) - \Psi(\rho e^{it})| \leqslant c_4 \, d(\Psi(e^{it}), \gamma_\rho), \tag{2.2}$$

where

$$\gamma_\rho = \{\zeta : |\Phi(\zeta)| = \rho\}.$$

Let

$$z_{n,k} = \Psi(e^{(2k\pi/n) i}), \qquad k = 1, 2, ..., n, \tag{2.3}$$

be $n$th Fejér points. We take $z_{n,n+1} = z_{n,1}$. By (2.1) and (2.2), we can find $c_0$, such that for $\Gamma_n = \gamma_{1 + c_0/n}$,

$$d(z_{n,k}, \Gamma_n) \leqslant \min_{j \neq k} |z_{n,j} - z_{n,k}|, \qquad k = 1, 2, ..., n, \tag{2.4}$$

and

$$|z_{n,k+1} - z_{n,k}| \leqslant c_5 \, d(z_{n,k}, \Gamma_n), \qquad k = 1, 2, ..., n, \tag{2.5}$$

hold.

Let $D_n$ be the interior of $\Gamma_n$. For $F \in L^p(\Gamma_n)$, we denote

$$\|F\|_{p,n} = \left\{ \int_{\Gamma_n} |F(z)|^p \, |dz| \right\}^{1/p}.$$

When $1 < p < \infty$, we define

$$\mathbb{P}F(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{F(\zeta)}{\zeta - z} \, d\zeta, \qquad z \in D_n;$$

then $\mathbb{P}F \in E^p(D_n)$, and [9]

$$\|\mathbb{P}F\|_p \leqslant c_6 \, \|F\|_{p,n}. \tag{2.6}$$

For $z \in \mathbb{C}$, $r > 0$, let

$$U(z, r) = \{\zeta : |\zeta - z| < r\}$$

and let

$$S(z, r) = \int_{\zeta \in \Gamma \cap U(z, r)} |d\zeta|.$$

Since $\Gamma$ is piecewise smooth, we have

$$S(z, r) \leqslant c_7 r. \tag{2.7}$$

Since $D$ is a Lipschitz domain, for any $z_1, z_2 \in \Gamma$, there exists an arc $\widehat{z_1 z_2} \subset \Gamma$ connecting $z_1$ and $z_2$, such that

$$|\widehat{z_1 z_2}| = \int_{\widehat{z_1 z_2}} |dz| \leqslant c_8 \, |z_1 - z_2|. \tag{2.8}$$

## 3. Uniform Separated

Points $\{w_j\}$ in $U$ are called $\eta_1$-uniformly separated, if

$$\inf_k \prod_{j \neq k} \frac{|w_j - w_k|}{|1 - \bar{w}_k w_j|} \geqslant \eta_1 > 0$$

and we call $\{w_j\}$ $\eta_2$-weakly separated, if

$$\inf_{j \neq k} \left| \frac{w_j - w_k}{1 - \bar{w}_k w_j} \right| \geqslant \eta_2 > 0, \qquad \text{for all } k.$$

Let $z_0$ be a fixed point in $D$, and let $\phi_n : D_n \to U$ be the conformal mapping satisfying $\phi_n(z_0) = 0$ and $\phi_n'(z_0) > 0$. We denote the inverse mapping of $\phi_n$ by $\psi_n$.

LEMMA 1. *For any* $n>0$, $\{\phi_n(z_{n,k}), k=1, 2, ..., n\}$ *are* $\frac{1}{16}$-*weakly separated.*

*Proof.* Let $\phi_{n,k}: D_n \to U$ be the conformal mapping satisfying $\phi_{n,k}(z_{n,k})=0$ and $\phi'_{n,k}(z_{n,k})>0$. Then [9, p. 96]

$$\phi'_{n,k}(z_{n,k})\, d(z_{n,k}, \Gamma_n) \geqslant \tfrac{1}{4}.$$

It is very easy to verify that [11]

$$\phi^*(w)=\frac{\phi_{n,k}(d(z_{n,k}, \Gamma_n)\, w+z_{k,n})}{\phi'_{n,k}(z_{n,k})\, d(z_{n,k}, \Gamma_n)} \in S.$$

By the Koebe's $\frac{1}{4}$-theorem, we know that $\{\phi_{n,k}(z): |z-z_{k,n}|<d(z_{n,k}, \Gamma_n)\}$ covers $\{w: |w|<\frac{1}{4}\phi'_{n,k}(z_{n,k})\, d(z_{n,k}, \Gamma_n)\}$.

By (2.4), for $j \neq k$,

$$\begin{aligned}\left|\frac{\phi_n(z_{n,j})-\phi_n(z_{n,k})}{1-\overline{\phi_n(z_{n,k})}\,\phi_n(z_{n,j})}\right| &= |\phi_{n,k}(z_{n,j})| \\ &\geqslant \frac{1}{4}\phi'_{n,k}(z_{n,k})\, d(z_{n,k}, \Gamma_n) \\ &\geqslant \frac{1}{16}. \quad \blacksquare\end{aligned}$$

A positive meausre $\mu$ on $D_n$ is called $\eta_3$-Carleson measure, if for any $z \in \Gamma_n$, $r>0$, we have

$$\mu(U(z, r)) \leqslant \eta_3 r.$$

Let $\delta_z$ be the unit mass concentrated at $z$.

LEMMA 2. *For any* $n>0$, *let*

$$v_n=\sum_{k=1}^{n} |z_{n,k+1}-z_{n,k}|\, \delta_{z_{n,k}}.$$

*Then* $v_n$ *is a* $c_9$-*Carleson measure on* $D_n$.

*Proof.* In fact, we only prove the lemma when $n$ is sufficiently large. For any $\zeta \in \Gamma_n$, $r>0$, there exists $\zeta^* \in \mathbb{C}\backslash D_n$ such that

$$|\zeta^*-\zeta|=r$$

and

$$d(\zeta^*, \Gamma_n) \geqslant c_{10} r. \tag{3.1}$$

For $z \in U(\zeta, r)$, we have

$$\frac{1}{r^2} \leqslant \frac{4}{|z-\zeta^*|^2}.$$

Then

$$\begin{aligned} v_n(U(\zeta, r)) &= \sum_{z_{n,k} \in U(\zeta, r)} |z_{n,k+1} - z_{n,k}| \\ &\leqslant 4r^2 \sum_{k=1}^{n} \frac{|z_{n,k+1} - z_{n,k}|}{|z_{n,k} - \zeta^*|^2} \end{aligned} \tag{3.2}$$

For any $z \in \widehat{z_{n,k} z_{n,k+1}}$, by (2.8) we have

$$\begin{aligned} |z - z_{n,k}| &\leqslant |\widehat{z z_{n,k}}| \\ &\leqslant |\widehat{z_{n,k} z_{n,k+1}}| \\ &\leqslant c_8 \, |z_{n,k+1} - z_{n,k}|. \end{aligned}$$

Then

$$\begin{aligned} |z - \zeta^*| &\leqslant |z - z_{n,k}| + |z_{n,k} - \zeta^*| \\ &\leqslant c_8 \, |z_{n,k+1} - z_{n,k}| + |z_{n,k} - \zeta^*| \\ &\leqslant c_8 \, d(z_{n,k}, \Gamma_n) + |z_{n,k} - \zeta^*| \\ &\leqslant (1 + c_8) \, |z_{n,k} - \zeta^*|. \end{aligned}$$

That means

$$\frac{1}{|z_{n,k} - \zeta^*|} \leqslant \frac{(1+c_8)}{|z-\zeta^*|}, \qquad z \in \widehat{z_{n,k} z_{n,k+1}}.$$

Hence

$$\begin{aligned} \sum_{k=1}^{n} \frac{|z_{n,k+1} - z_{n,k}|}{|z_{n,k} - \zeta^*|^2} &\leqslant (1+c_8)^2 \sum_{k=1}^{n} \int_{\widehat{z_{n,k} z_{n,k+1}}} \frac{|dz|}{|z-\zeta^*|^2} \\ &= (1+c_8)^2 \int_{\Gamma} \frac{|dz|}{|z-\zeta^*|^2} \\ &= (1+c_8)^2 \int_{d(\zeta^*, \Gamma)}^{+\infty} \frac{dS(\zeta^*, t)}{t^2} \\ &\leqslant c_{11} \frac{1}{d(\zeta^*, \Gamma)}. \end{aligned}$$

The last inequality is because of (2.7). By (3.1) and (3.2) we have

$$\nu_n(U(\zeta, r)) \leqslant c_{12} r. \quad \blacksquare$$

LEMMA 3. *For any* $n > 0$, $\{\phi_n(z_{n,k}), k = 1, 2, ..., n\}$ *are* $c_{13}$*-uniformly separated.*

*Proof.* Since we have shown that $\{\phi_n(z_{n,k}), k = 1, 2, ..., n\}$ are $\frac{1}{16}$-weakly separated, we only need prove that

$$\mu_n = \sum_{k=1}^{n} (1 - |\phi_n(z_{n,k})|^2)\, \delta_{\phi_n(z_{n,k})}$$

is a $c_{14}$-Carleson measure on $U$ [10, p. 287].

Since $\nu_n$ is a $c_{10}$-Carleson measure on $D_n$, then for any $h \in E^1(D_n)$, we have [9]

$$\sum_{k=1}^{n} |h(z_{n,k})|\, |z_{n,k+1} - z_{n,k}| = \iint_{D_n} |h|\, d\nu_n$$
$$\leqslant c_{15} \|h\|_{1,n}.$$

Let

$$g(w) = \psi_n'(w)\, h \circ \psi_n(w), \qquad w \in U, \tag{3.3}$$

then $g \in H^1$, and

$$\sum_{k=1}^{n} |g \circ \phi_n(z_{n,k})|\, |\phi_n'(z_{n,k})|\, |z_{n,k+1} - z_{n,k}| \leqslant c_{15} \|g\|_1.$$

By the Koebe distortion theorem [9, p. 96] and (2.5)

$$\begin{aligned} 1 - |\phi_n(z_{n,k})|^2 &\leqslant 8\, |\phi_n'(z_{n,k})|\, d(z_{n,k}, \Gamma_n) \\ &\leqslant 8\, |\phi_n'(z_{n,k})|\, |z_{n,k+1} - z_{n,k}|. \end{aligned} \tag{3.4}$$

Then

$$\iint_U |g|\, d\mu_n \leqslant 8c_{15} \|g\|_1.$$

Since very function in $H^1$ can be written in the form of (3.3), the above inequality holds for any $g \in H^1$, which is equivalent to that $\mu_n$ is a $c_{14}$-Carleson measure on $U$. $\blacksquare$

LEMMA 4. *Suppose* $0<p<\infty$, $\{a_k, k=1, ..., n\}$ *are complex numbers. There exists* $h \in E^p(D_n)$ *such that*

$$h(z_{n,k})=a_k, \qquad k=1,2,...,n, \tag{3.5}$$

*and*

$$\|h\|_{p,n} \leqslant c_{16}\left\{\sum_{k=1}^{n} |a_k|^p\,|z_{n,k+1}-z_{n,k}|\right\}^{1/p}$$

*Proof.* Let

$$b_k=a_k[\phi_n'(z_{n,k})]^{-1/p}, \qquad k=1,2,...,n.$$

From Lemma 3 we can find a $g \in H^p$ such that [10]

$$(g\circ\phi_n)(z_{n,k})=b_k, \qquad k=1,2,...,n,$$

and

$$\|g\|_p \leqslant c_{16}\left\{\sum_{k=1}^{n} |b_k|^p\,(1-|\phi_n(z_{n,k})|^2)\right\}^{1/p}.$$

Let

$$h(z)=[\phi_n'(z)]^{1/p}\,(g\circ\phi_n)(z)\in E^p(D_n).$$

Then we have (3.5). By (3.4) we have

$$\|h\|_{p,n}=\|g\|_p \leqslant 8^{1/p}c_{16}\left\{\sum_{k=1}^{n} |a_k|^p\,|z_{n,k+1}-z_{n,k}|\right\}^{1/p}. \quad \blacksquare$$

## 4. AN ESTIMATION OF $|\omega_n(z)|$ ON $\Gamma_n$

Let

$$\omega_n(z)=\prod_{k=1}^{n}(z-z_{n,k}).$$

LEMMA 5. *For any* $z\in\Gamma_n$

$$c_{17}^{-1}\leqslant\left|\frac{\omega_n(z)}{d^n}\right|\leqslant c_{17}, \tag{4.1}$$

*where* $d=\Psi'(\infty)$.

*Proof.* As in [4], the function

$$\chi(w, u)=\begin{cases}\dfrac{\Psi(w)-\Psi(u)}{d(w-u)}, & u \neq w \\ \dfrac{\Psi'(w)}{d}, & u=w\end{cases}$$

is clearly an analytic function of $u$ for fixed $w$, $|u|>1$, $|w|>1$, and $\chi(w, \infty)=1$. The univalence of $\Psi(w)$ implies that $\chi(w, u)$ cannot vanish for $|u|>1$, $|w|>1$.

Let $\log \chi(w, u)$ denote the branch of logarithm for which $\log \chi(w, \infty)=0$; then we have the Laurent series

$$\log \chi(w, u)=\sum_{j=1}^{\infty} \frac{a_j(w)}{u^j}.$$

For $z=\Psi(w) \in \Gamma_n$, we have

$$\log \frac{\omega_n(z)}{d^n(w^n-1)}=n \sum_{l=1}^{+\infty} a_{nl}(w). \tag{4.2}$$

For $|w|=1+c_0/n$, $k \geqslant n$, we now estimate $|a_k(w)|$. Evidently

$$\begin{aligned} a_k(w) &= \frac{1}{2k(k+1)\pi i} \int_{|u|=1+c_0/2k} u^{k+1} \frac{\partial^2 \log \chi(w, u)}{\partial u^2}\, du \\ &= \frac{1}{2k(k+1)\pi i} \int \frac{u^{k+1}\, du}{(u-w)^2} - \frac{1}{2k(k+1)\pi i} \int \frac{u^{k+1}[\Psi'(u)]^2}{[\Psi(u)-\Psi(w)]^2}\, du \\ &\quad + \frac{1}{2k(k+1)\pi i} \int \frac{u^{k+1}\Psi''(u)}{\Psi(u)-\Psi(w)}\, du \\ &= B_1(w)+B_2(w)+B_3(w). \end{aligned} \tag{4.3}$$

For the sake of simplicity we omit the path of integration $|u|=1+c_0/2k$ in the following part of this section. There is no essential effect and notations and computation are much easier if we assume that there is only one corner on $\Gamma$, $\tau_1=1$ and $\alpha_1=\alpha$.

Since $|u|<|w|$, we have

$$B_1(w)=0. \tag{4.4}$$

By (1.3) we have

$$|B_2(w)| \leqslant \frac{(1+c_0/2k)^{k+1}}{2k(k+1)\pi} \int \frac{|\lambda(u)|\,|u-1|^{2\alpha-2}}{|\Psi(u)-\Psi(w)|^2}\,|du|$$
$$\leqslant \frac{c_{19}}{k^2} \int \frac{|u-1|^{2\alpha-2}\,|du|}{|u-w|^2\,(|u-1|+|u-w|)^{2\alpha-2}}.$$

If $\alpha \geqslant 1$, clearly we have

$$|B_2(w)| \leqslant \frac{c_{19}}{k^2} \int \frac{|du|}{|u-w|^2} \leqslant \frac{c_{20}n}{k^2}.$$

In the case when $0<\alpha<1$, we have

$$|B_2(w)| \leqslant \frac{2c_{19}}{k^2} \int \frac{|u-1|^{2-2\alpha}+|u-w|^{2-2\alpha}}{|u-w|^2\,|u-1|^{2-2\alpha}}\,|du|$$
$$\leqslant \frac{2c_{19}}{k^2} \int \frac{|du|}{|u-w|^2} + \frac{2c_{19}}{k^2} \int \frac{|du|}{|u-w|^{2\alpha}\,|u-1|^{2-2\alpha}}$$
$$\leqslant \frac{2c_{20}n}{k^2} + \frac{2c_{19}}{k^2} \left\{\int \frac{|du|}{|u-w|^2}\right\}^{\alpha} \left\{\int \frac{|du|}{|u-1|^2}\right\}^{1-\alpha}.$$

The last inequality is because of Hölder's inequality. Hence

$$|B_2(w)| \leqslant \frac{2c_{20}n}{k^2} + \frac{c_{21}n^{\alpha}}{k^{1+\alpha}} \tag{4.5}$$

holds in both cases $1 \leqslant \alpha < 2$ and $0<\alpha<1$.

By (1.3) we have

$$\Psi''(u) = -\frac{(\alpha-1)\,\lambda(u)}{u^2}(u^{-1}-1)^{\alpha-2} + \lambda'(u)(u^{-1}-1)^{\alpha-1}.$$

Hence

$$|B_3(w)| \leqslant \frac{c_{22}}{k^2} \int \frac{|\lambda(u)|\,|u-1|^{\alpha-2}}{|\Psi(u)-\Psi(w)|}\,|du|$$
$$+\frac{c_{22}}{k^2} \int \frac{|\lambda'(u)|\,|u-1|^{\alpha-1}}{|\Psi(u)-\Psi(w)|}\,|du|$$
$$= B_{31}(w) + B_{32}(w). \tag{4.6}$$

By (2.1) we have

$$B_{31}(w) \leqslant \frac{c_{23}}{k^2} \int \frac{|u-1|^{\alpha-2}\,|du|}{|u-w|\,(|u-1|+|u-w|)^{\alpha-1}}.$$

If $\alpha \geqslant 1$ we have

$$\begin{aligned} B_{31} &\leqslant \frac{c_{23}}{k^2} \int \frac{|du|}{|u-w|\,|u-1|} \\ &\leqslant \frac{c_{23}}{k^2} \left\{ \int \frac{|du|}{|u-w|^2} \right\}^{1/2} \left\{ \int \frac{|du|}{|u-1|^2} \right\}^{1/2} \\ &\leqslant \frac{c_{24} n^{1/2}}{k^{3/2}}. \end{aligned}$$

In the case when $0 < \alpha < 1$

$$\begin{aligned} B_{31}(w) &\leqslant \frac{c_{23}}{k^2} \int \frac{|u-1|^{1-\alpha} + |u-w|^{1-\alpha}}{|u-w|\,|u-1|^{2-\alpha}}\,|du| \\ &= \frac{c_{23}}{k^2} \int \frac{|du|}{|u-w|\,|u-1|} + \frac{c_{23}}{k^2} \int \frac{|du|}{|u-1|^{2-\alpha}} \\ &\leqslant \frac{c_{24} n^{1/2}}{k^{3/2}} + \frac{c_{25} n^{\alpha}}{k^2} \int \frac{|du|}{|u-1|^{2-\alpha}} \\ &\leqslant \frac{c_{24} n^{1/2}}{k^{3/2}} + \frac{c_{26} n^{\alpha}}{k^{1+\alpha}}. \end{aligned}$$

Hence

$$B_{31} \leqslant \frac{c_{24} n^{1/2}}{k^{3/2}} + \frac{c_{26} n^{\alpha}}{k^{1+\alpha}} \tag{4.7}$$

holds in both cases $1 \leqslant \alpha < 2$ and $0 < \alpha < 1$.

Since $\lambda(u) \in \mathrm{Lip}_\beta$ we have [12, p. 74]

$$|\lambda'(u)| \leqslant c_{27}(1-|u^{-1}|)^{\beta-1}, \qquad |u| \geqslant 1.$$

Hence

$$B_{32} \leqslant \frac{c_{28}}{k^{1+\beta}} \int \frac{|u-1|^{\alpha-1}}{|u-w|\,(|u-1|+|u-w|)^{\alpha-1}}\,|du|.$$

If $\alpha \geqslant 1$, we have

$$B_{32}(w) \leqslant \frac{c_{28}}{k^{1+\beta}} \int \frac{|du|}{|u-w|} \leqslant c_{29} \frac{\log n}{k^{1+\beta}}.$$

In the case when $0<\alpha<1$, we have

$$\begin{aligned} B_{32}(w) &\leqslant \frac{c_{28}}{k^{1+\beta}} \int \frac{|du|}{|u-w|} + \frac{c_{28}}{k^{1+\beta}} \int \frac{|du|}{|u-w|^{\alpha} |u-1|^{1-\alpha}} \\ &\leqslant \frac{c_{29} \log n}{k^{1+\beta}} + \frac{c_{28}}{k^{1+\beta}} \left\{ \int \frac{|du|}{|u-w|} \right\}^{\alpha} \left\{ \int \frac{|du|}{|u-1|} \right\}^{1-\alpha} \\ &\leqslant \frac{c_{30} \log k}{k^{1+\beta}}. \end{aligned}$$

Then we always have

$$B_{32} \leqslant \frac{c_{30} \log k}{k^{1+\beta}}, \qquad 0<\alpha<2. \tag{4.8}$$

Combining (4.3)–(4.8) we conclude

$$|a_k(w)| \leqslant c_{31} \left( \frac{n^{1/2}}{k^{3/2}} + \frac{n^{\alpha}}{k^{1+\alpha}} + \frac{\log k}{k^{1+\beta}} \right).$$

Together with (4.2) we have

$$\left| \log \frac{\omega_n(z)}{d^n(w^n-1)} \right| \leqslant c_{32}, \qquad z = \Psi(w), \quad |w| = 1 + \frac{c_0}{n}.$$

That implies (4.1). ∎

## 5. Marcinkiewicz–Zygmund Inequalities

We extend the Marcinkiewicz–Zygmund inequalities to the admissible domain.

Lemma 6. *Suppose* $1<p<\infty$; *then for any* $P_{n-1}$, *a polynomial of degree at most* $n-1$, *we have*

$$\|P_{n-1}\|_p \leqslant c_{33} \left\{ \sum_{k=1}^{n} |P_{n-1}(z_{n,k})|^p \, |z_{n,k+1} - z_{n,k}| \right\}^{1/p}.$$

*Remark.* By the Bernstein inequality we know that $\|P_{n-1}\|_{p,n} \leqslant c_{34} \|P_{n-1}\|_p$; from lemma 2 we can easily get

$$\left\{ \sum_{k=1}^{n} |P_{n-1}(z_{n,k})|^p \, |z_{n,k+1} - z_{n,k}| \right\}^{1/p} \leqslant c_{35} \|P_{n-1}\|_p.$$

This is the other part of Marcinkiewicz–Zygmund inequalities.

*Proof.* From Lemma 4, there exists an $h \in E^p(D_n)$, such that

$$h(z_{n,k}) = P_{n-1}(z_{n,k})$$

and

$$\|h\|_{p,n} \leqslant c_{16} \left\{ \sum_{k=1}^{n} |P_{n-1}(z_{n,k})|^p \, |z_{n,k+1} - z_{n,k}| \right\}^{1/p} \tag{5.1}$$

Since $P_{n-1}(z)$ is the Lagrange interpolation polynomial to $h(z)$ at $\{z_{n,k}\}$, we have

$$P_{n-1}(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\omega_n(\zeta) - \omega_n(z)}{\omega_n(\zeta)} \frac{h(\zeta)}{\zeta - z} \, d\zeta.$$

For $z \in D_n$, we have

$$\begin{aligned} h(z) - P_{n-1}(z) &= \frac{\omega_n(z)}{2\pi i} \int \frac{f(\zeta)}{\omega_n(\zeta)} \frac{d\zeta}{\zeta - z} \\ &= \omega_n(z) \, \mathbb{P}\left(\frac{f}{\omega_n}\right)(z), \qquad z \in D_n. \end{aligned}$$

By (2.6) and Lemma 5

$$\begin{aligned} \|h - P_{n-1}\|_p &\leqslant \max_{z \in \Gamma_n} |\omega_n(z)| \left\| \mathbb{P}\left(\frac{h}{\omega_n}\right) \right\|^p \\ &\leqslant c_6 \max_{z \in \Gamma_n} |\omega_n(z)| \left\| \frac{h}{\omega_n} \right\|_{p,n} \\ &\leqslant c_6 \max_{\zeta, z \in \Gamma_n} \left| \frac{\omega_n(z)}{\omega_n(\zeta)} \right| \|h\|_{p,n} \\ &\leqslant c_6 c_{17}^2 \|h\|_{p,n}. \end{aligned}$$

Since $\mathbb{P}h = h$, we also have $\|h\|_p \leqslant c_6 \|h\|_{p,n}$. Then

$$\|P_{n-1}\|_p \leqslant c_{36} \|h\|_{p,n}.$$

And by (5.1), we completed the proof of Lemma 6. ∎

## 6. Proof of the Theorem

It is sufficient to show that

$$\|L_n(f, z)\|_p \leqslant c_{37} \max_{z \in \bar{D}} |f(z)|$$

holds for $1 < p < \infty$, $f \in A(\bar{D})$.

From Lemma 6

$$\begin{aligned}\|L_n(f, z)\|_p &\leqslant c_{33} \left\{ \sum_{k=1}^{n} |f(z_{n,k})|^p \, |z_{n,k+1} - z_{n,k}| \right\}^{1/p} \\ &\leqslant c_{33} \, |\Gamma|^{1/p} \max_{z \in \bar{D}} |f(z)|,\end{aligned}$$

where $|\Gamma|$ means the length of $\Gamma$. This completes the proof of the theorem.

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